MIXED TYPE SURFACES WITH BOUNDED MEAN CURVATURE IN 3-DIMENSIONAL SPACE-TIMES

A. HONDA, M. KOISO, M. KOKUBU, M. UMEHARA, AND K. YAMADA

Abstract. In this paper, we shall prove that space-like surfaces with bounded mean curvature functions in real analytic Lorentzian 3-manifolds can change their causality to time-like surfaces only if the mean curvature functions tend to zero. Moreover, we shall show the existence of such surfaces with non-vanishing mean curvature and investigate their properties.

1. Introduction

We say that a connected surface immersed in a Lorentzian 3-manifold (M^3, g) is of *mixed type* if both the space-like and time-like point sets are non-empty. In general, the mean curvature of such surfaces diverges: for example, the graph of a smooth function $t = f(x, y)$ in the Lorentz-Minkowski space-time $(\boldsymbol{R}^3_1; t, x, y)$ gives a space-like (resp. time-like) surface if $B > 0$ (resp. $B < 0$), where

$$B := 1 - f_x^2 - f_y^2. \tag{1.1}$$

In this situation, the unit normal vector is given by

$$\nu = \frac{1}{\sqrt{|B|}}(1, f_x, f_y), \tag{1.2}$$

and the mean curvature function is computed as

$$H = \frac{\left(f_x^2 - 1\right) f_{yy} - 2 f_x f_y f_{xy} + \left(f_y^2 - 1\right) f_{xx}}{2|B(x,y)|^{3/2}}, \tag{1.3}$$

which is unbounded around the set $\{B(x, y) = 0\}$, in general.

On the other hand, several zero mean curvature surfaces of mixed type in $\boldsymbol{R}^3_1$ were found in [11], [7], [10], [12], [9], [5], [2] and [3]. Moreover, such examples can be found in other space-times: in fact, a zero mean curvature surface of mixed type in the de Sitter 3-space (resp. in the anti-de Sitter 3-space) is given in this paper (cf. Example 2.6 and Example 2.7). It is known that zero mean curvature surfaces in $\boldsymbol{R}^3_1$ change types across their fold singularities, except for the special case as in [2]. On the other hand, in [8], it was shown that space-like non-zero constant mean curvature surfaces do not admit fold singularities, which suggests that space-like non-zero constant mean curvature surfaces never change types. More precisely, the following questions naturally arise:

(a) *Is there a mixed type surface with non-zero constant mean curvature?*

Date: August 12, 2015.

2010 *Mathematics Subject Classification.* Primary:53A35; Secondary 57R42, 35M10.

Key words and phrases. causality, type change, mean curvature, Lorentzian manifolds .

(b) *Is there a mixed type surface whose mean curvature vector field is smooth and does not vanish along the curve of type change?*

In this paper, we show that the answer to Question (a) is negative. This is a consequence of the following assertion:

Theorem 1.1. *Let U be a connected domain in $\boldsymbol{R}^2$, and $f : U \to (M^3, g)$ a real analytic immersion into an oriented real analytic Lorentzian manifold (M^3, g). We denote by U_+ (resp. U_-) the set of points where f is space-like (resp. time-like). Suppose that U_+, U_- are both non-empty, and the mean curvature function H on $U_+ \cup U_-$ is bounded. Then for each $p \in \overline{U_+} \cap \overline{U_-}$, there exists a sequence $\{p_n\}_{n=1,2,3,..}$ in U_+ (resp. U_-) converging to p so that $\lim_{n\to\infty} H(p_n) = 0$, where $\overline{U_+}, \overline{U_-}$ are the closures of U_+, U_- in U.*

There exist space-like and time-like constant mean curvature immersions in $\boldsymbol{R}^3_1$ which are not of mixed type although their induced metrics degenerate along certain smooth curves (cf. Examples 2.3 and 2.4 in Section 2). Also, there are similar such examples of space-like constant mean curvature one surfaces in the de Sitter 3-space S^3_1 with singularities which are not of mixed type ([1]). The existence of such examples implies that we cannot drop the assumption that both U_+, U_- are non-empty. The proof of Theorem 1.1 is given in Section 2.

On the other hand, we show that the answer to Question (b) is affirmative. In fact, we show in Section 3 that the mean curvature vector fields of real analytic surfaces of mixed type with bounded mean curvature functions can be analytically extended across the sets of type change under a suitable genericity assumption (cf. Proposition 3.6). Moreover, we show the following:

Theorem 1.2. *There exists a real analytic function $g(x, y)$ on $\boldsymbol{R}^2$ whose graph realized in $\boldsymbol{R}^3_1$ satisfies the following properties:*

1. *the set Σ_g of non-degenerate points of type change of the graph of g is non-empty, and the induced metric of the graph of g is non-degenerate on $\boldsymbol{R}^2 \setminus \Sigma_g$,*
2. *the mean curvature function of the graph of g is bounded on $\boldsymbol{R}^2 \setminus \Sigma_g$.*
3. *the mean curvature vector field can be extended to Σ_g real analytically, and does not vanish at each point of Σ_g.*

This suggests that surfaces with smooth mean curvature vector fields form an important sub-class of the set of mixed type surfaces.

2. Behavior of mean curvature along curves of type change

Let (M^3, g) be an oriented real analytic Lorentzian 3-manifold. Then, the vector product $\boldsymbol{v} \times_g \boldsymbol{w}$ is defined for linearly independent tangent vectors $\boldsymbol{v}, \boldsymbol{w}$ at $p \in M^3$, satisfying the following three properties:

1. $\boldsymbol{v} \times_g \boldsymbol{w}$ is orthogonal to $\boldsymbol{v}$ and $\boldsymbol{w}$,
2. $\{\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{v} \times_g \boldsymbol{w}\}$ is a basis of the tangent space T_pM which is compatible with the orientation of M^3,
3. it holds that

$$g_p(\boldsymbol{v} \times_g \boldsymbol{w}, \boldsymbol{v} \times_g \boldsymbol{w}) = -g_p(\boldsymbol{v}, \boldsymbol{v})g_p(\boldsymbol{w}, \boldsymbol{w}) + g_p(\boldsymbol{v}, \boldsymbol{w})^2.$$

For each tangent vector $\boldsymbol{v} \in T_pM^3$ ($p \in M^3$), we set

$$|\boldsymbol{v}| := \sqrt{|g_p(\boldsymbol{v}, \boldsymbol{v})|}.$$

We fix a domain U in $\boldsymbol{R}^2$. Let $f : U \to M^3$ be a real analytic immersion. Set $f_u := df(\partial_u)$, $f_v := df(\partial_v)$, where $\partial_u := \partial/\partial u$, $\partial_v := \partial/\partial v$. Using three real analytic functions

$$g_{11} := g(f_u, f_u), \quad g_{12} = g_{21} = g(f_u, f_v), \quad g_{22} := g(f_v, f_v)$$

on U, we define a function $\beta : U \to \boldsymbol{R}$ by

$$\beta := g_{11}g_{22} - g_{12}^2. \tag{2.1}$$

Then

$$U_+ := \{p \in U \,;\, \beta(p) > 0\}, \qquad U_- := \{p \in U \,;\, \beta(p) < 0\}$$

give the set of space-like points and the set of time-like points, respectively. The unit normal vector field

$$\nu = \frac{f_u \times_g f_v}{|f_u \times_g f_v|} \tag{2.2}$$

of f is well-defined on $U_+ \cup U_-$. Using this, we set

$$h_{11} := g(f_{uu}, \nu), \quad h_{12} = h_{21} = g(f_{uv}, \nu), \quad h_{22} := g(f_{vv}, \nu),$$

where

$$f_{uu} = \nabla_{\partial_u} f_u, \quad f_{uv} = \nabla_{\partial_v} f_u = \nabla_{\partial_u} f_v, \quad f_{vv} = \nabla_{\partial_v} f_v,$$

and ∇ is the Levi-Civita connection of the Lorentzian manifold (M^3, g). Each h_{ij} ($i, j = 1, 2$) is a function defined on $U_+ \cup U_-$. The mean curvature function H is also defined on $U_+ \cup U_-$, and is given by

$$H := \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{2|\beta|} = \frac{\alpha}{2|\beta|^{3/2}}, \tag{2.3}$$

where

$$\alpha := \sqrt{|\beta|}(g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}). \tag{2.4}$$

Then the following assertion holds:

Lemma 2.1. *The function $\alpha : U_+ \cup U_- \to \boldsymbol{R}$ can be analytically extended to U.*

Proof. We set $\tilde{\nu} := f_u \times_g f_v$. Then

$$\beta = -g(f_u \times_g f_v, f_u \times_g f_v)$$

and $\nu = \tilde{\nu}/\sqrt{|\beta|}$ holds (cf. (2.2)). Therefore, we have that

$$\begin{aligned} \alpha &= \sqrt{|\beta|}\Big(g(f_{vv}, \nu)g_{11} - 2g(f_{uv}, \nu)g_{12} + g(f_{uu}, \nu)g_{22}\Big) \\ &= g(f_{vv}, \tilde{\nu})g_{11} - 2g(f_{uv}, \tilde{\nu})g_{12} + g(f_{uu}, \tilde{\nu})g_{22}, \end{aligned}$$

proving the assertion. $\square$

Using the lemma, we now give the proof of Theorem 1.1:

Proof of Theorem 1.1. We may assume that the mean curvature function H is not identically zero. Let (x^1, x^2) be the coordinates of U. We fix a point $p \in \overline{U_+} \cap \overline{U_-}$. Let $\varepsilon > 0$ be an arbitrary positive number and V a neighborhood of p. It is sufficient to show that there exist points $q_+ \in V_+$ and $q_- \in V_-$ such that $|H(q_+)|$ and $|H(q_-)|$ are both less than ε. We may assume that V is connected. If $\beta \geq 0$ or $\beta \leq 0$ on V, this contradicts the fact that $p \in \overline{U_+} \cap \overline{U_-}$. So, we can take two points $q_0, q_1 \in V$ such that $\beta(q_0) > 0$ and $\beta(q_1) < 0$. We then take a smooth curve $\gamma(s)$ $(s \in [0, 2\pi])$ on V such that $\gamma(0) = q_0$ and $\gamma(2\pi) = q_1$. Since the image of γ lies in V, we can write $\gamma = (\gamma^1, \gamma^2)$ and each γ^i $(i = 1, 2)$ has the following Fourier series expansion:

$$\gamma^i(s) = u_0^i + \sum_{k=1}^{\infty} \left(u_k^i \cos ks + v_k^i \sin ks \right) \qquad (i = 1, 2).$$

We then set

$$\gamma_N^i(s) = u_0^i + \sum_{k=1}^{N} \left(u_k^i \cos ks + v_k^i \sin ks \right) \qquad (i = 1, 2),$$

where N is a sufficiently large positive integer. Then the real analytic curve defined by $\gamma_N(s) := (\gamma_N^1(s), \gamma_N^2(s))$ satisfies

$$\beta(\gamma_N(0)) > 0, \qquad \beta(\gamma_N(2\pi)) < 0. \tag{2.5}$$

Since

$$\hat{\beta}(s) := \beta(\gamma_N(s)) \qquad (0 \leq s \leq 2\pi)$$

is a real analytic function defined on $[0, 2\pi]$, the set of zeros of the function $\hat{\beta}(s)$ consists of a finite set of points

$$0 < s_1 < \cdots < s_n < 2\pi.$$

By (2.5), we can choose the number j such that the sign of $\hat{\beta}(s)$ changes from positive to negative at $s = s_j$. Then there exists a positive integer m such that

$$\lim_{s \to s_j} \frac{\hat{\beta}(s)}{(s - s_j)^m} = b \, (\neq 0),$$

where b is a non-zero real number. Since $\hat{\beta}(s)$ changes sign at $s = s_j$, the integer m is odd. By Lemma 2.1, we may regard α as a real analytic function on U. So we set

$$\hat{\alpha}(s) := \alpha(\gamma_N(s)).$$

By (2.3), we have that

$$H(\gamma_N(s)) := \frac{\hat{\alpha}(s)}{2|\hat{\beta}(s)|^{3/2}}$$

for $s \neq s_1, ..., s_n$. Since H is bounded, we have $\hat{\alpha}(s_j) = 0$. Since $\hat{\alpha}(s)$ is a real analytic function, there exists a positive integer ℓ such that

$$\lim_{s \to s_j} \frac{\hat{\alpha}(s)}{(s - s_j)^\ell} = a \, (\neq 0),$$

where a is a non-zero real number. Then it holds that

$$\lim_{s \to s_j} |s - s_j|^{(3m/2) - \ell} |H(\gamma_N(s))| = \frac{|a|}{|b|^{3/2}} \, (\neq 0).$$

Since H is bounded, we have $2\ell \geq 3m$. Moreover, since m is odd, we have $2\ell > 3m$. Then we have $\lim_{s\to s_j} |H(\gamma_N(s))| = 0$. In particular, if we set

$$q_+ := \gamma_N(s_j - \delta), \qquad q_- := \gamma_N(s_j + \delta),$$

then $|H(q_+)|$ and $|H(q_-)|$ are less than ε for sufficiently small $\delta > 0$. So we get the assertion. $\square$

As a consequence, we get the following corollary:

Corollary 2.2. *Under the assumption of Theorem 1.1, the function $\alpha : U_+ \cup U_- \to \boldsymbol{R}$ can be analytically extended to U and vanishes on $\overline{U_+} \cap \overline{U_-}$.*

Proof. By Lemma 2.1, the function α can be analytically extended to U. Suppose that $\alpha(p) \neq 0$ for $p \in \overline{U_+} \cap \overline{U_-}$. Then the mean curvature function cannot be bounded, since $\beta(p) = 0$. $\square$

We give here several examples:

Example 2.3 (A space-like CMC surface with parabolic symmetry)**.** Consider the map $f_P : \boldsymbol{R}^2 \to \boldsymbol{R}^3_1$ such that

$$f_P(u,v) := \left(-\eta(v) + u^2 v + v, -\eta(v) + u^2 v - v, 2uv\right),$$

where

$$\eta(v) := \frac{1}{2}\left(\arctan(v) - \frac{v}{v^2+1}\right), \qquad |\arctan(v)| < \frac{\pi}{2}.$$

This surface has singularities on the u-axis. Moreover, the inverse image $f_P^{-1}(\{\mathbf{0}\})$ coincides with the u-axis, where $\mathbf{0} := (0,0,0)$. One can easily check that f_P gives a space-like immersion of constant mean curvature $1/2$ on $\boldsymbol{R}^2 \setminus \{v = 0\}$. Moreover, the image of f_P is contained in the set (cf. Figure 1, left)

$$\mathcal{P} := \left\{(t,x,y) \in \boldsymbol{R}^3_1\,;\, -t^2 + x^2 + y^2 = 2(t-x)\eta\left(\frac{t-x}{2}\right)\right\}.$$

The light-like line

$$L := \{(c,c,0)\,;\, c \in \boldsymbol{R}\} = \left\{\lim_{u\to\infty} f_P(u, \frac{c}{u^2})\,;\, c \in \boldsymbol{R}\right\}$$

is contained in $\mathcal{P}$, and the image of f coincides with $\mathcal{P} \setminus L$. The set $\mathcal{P}$ itself is a surface in $\boldsymbol{R}^3_1$ without self-intersections which has a cone-like singular point at the origin $\mathbf{0}$, and has bounded mean curvature function on $\mathcal{P} \setminus \{\mathbf{0}\}$. Moreover, the induced metric on $\mathcal{P}$ degenerates only on the line L. This implies that we cannot drop the assumption that U_+, U_- are non-empty in the statement of Theorem 1.1. This example is an analogue of the maximal surface called *the Enneper surface of the 2nd kind* or *parabolic catenoid* (cf. [11], [2]).

Example 2.4 (A space-like CMC surface with hyperbolic symmetry)**.** We next consider the map defined by

$$f_H(u,v) := (v\cosh u, v \sinh u, \varphi(v)) \qquad ((u,v) \in \boldsymbol{R} \times (-1,1)),$$

where

$$\varphi(v) := \log\left(\frac{1+v}{1-v}\right) - v.$$

Like the case of f_P, this surface has singularities on the u-axis and $f_H^{-1}(\{\mathbf{0}\})$ coincides with the u-axis. One can easily check that f_H gives a space-like immersion

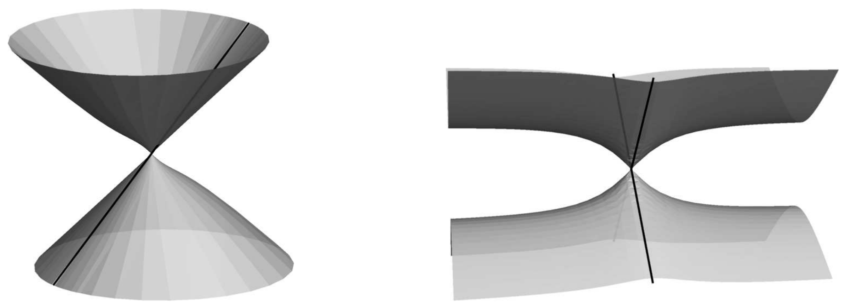

Figure 1. The figure of $\mathcal{P}$ (left) and $\mathcal{H}$ (right).

of constant mean curvature $1/2$ on $\boldsymbol{R}^2 \setminus \{v = 0\}$. Moreover, the image of f_H is contained in the set (cf. Figure 1, right)

$$\mathcal{H} := \left\{(t, x, y) \in \boldsymbol{R}^3_1 \,;\, y = \varphi(\pm\sqrt{t^2 - x^2})\right\} = \left\{(t, x, y) \in \boldsymbol{R}^3_1 \,;\, t^2 = x^2 + \psi(y)^2\right\},$$

where $\psi : \boldsymbol{R} \to (-1, 1)$ is the inverse function of $\varphi : (-1, 1) \to \boldsymbol{R}$. Two light-like lines

$$L_\pm := \{(c, \pm c, 0) \,;\, c \in \boldsymbol{R}\}$$

are contained in $\mathcal{H}$ and

$$\mathcal{H} = L_+ \cup L_- \cup (\text{Image of } f_H) \cup (\text{Image of } f'_H),$$

where

$$f'_H(u, v) := (-v \cosh u, v \sinh u, \varphi(v)) \qquad ((u, v) \in \boldsymbol{R} \times (-1, 1)).$$

Like as in the case of $\mathcal{P}$, the set $\mathcal{H}$ has no self-intersections, and has bounded mean curvature function on $\mathcal{H} \setminus \{\mathbf{0}\}$. The origin $\mathbf{0}$ is a cone-like singular point. Moreover, its induced metric degenerates along the lines $L_\pm$. This example is an analogue of the maximal surface called *the catenoid of the 2nd kind* or *hyperbolic catenoid* (cf. [11], [2]).

Similar examples, that is, a family of space-like surfaces with constant mean curvature one containing light-like lines in the de Sitter 3-space S^3_1 have recently been found in [1].

The following is one typical mixed type surface whose mean curvature vanishes identically.

Example 2.5. Consider the function

$$f_K(x, y) := x \tanh y.$$

Then the graph of f_K in $\boldsymbol{R}^3_1$ gives a zero mean curvature surface, which is space-like on the set $U_+ := \{(x, y) \in \boldsymbol{R}^2 \,;\, x^2 > \cosh^2 y\}$ and time-like on the set $U_- := \{(x, y) \in \boldsymbol{R}^2 \,;\, x^2 < \cosh^2 y\}$. This example is called the *helicoid of the 2nd kind*, which was found by Kobayashi [11].

On the other hand, we can find a similar example in another space form:

Example 2.6. Consider the map $f_Z : \boldsymbol{R} \times S^1 \to S^3_1$ given by

$$f_Z(u, v) := (\sinh u \sin v,\ \cos u \cos v,\ \sin u \cos v,\ \cosh u \sin v)\,,$$

where

$$S^3_1 := \{(t,x,y,z) \in \boldsymbol{R}^4_1\,;\, -t^2+x^2+y^2+z^2=1\}$$

is the de Sitter 3-space, which is the space-time of constant sectional curvature 1. Then the first fundamental form of f_Z is given by $ds^2 = \cos 2v\, du^2 + dv^2$. In particular, f_Z is space-like (resp. time-like) if $\cos 2v > 0$ (resp. $\cos 2v < 0$). Moreover, the mean curvature function of f_Z vanishes identically.

Example 2.7. We define an immersion $f_{\mathrm{ads}} : \boldsymbol{R}^2 \to H^3_1$ by

$$f_{\mathrm{ads}}(u,v) = (\cosh u \cosh v,\, \sinh au \sinh v,\, \cosh au \sinh v,\, \sinh u \cosh v)\,,$$

where $a = 1/\tanh\alpha$ ($\alpha \neq 0$) is a constant, and

$$H^3_1 = \left\{(t,x,y,z) \in \boldsymbol{R}^4_2\,;\, -t^2-x^2+y^2+z^2=-1\right\}$$

is the anti-de Sitter 3-space, which is the space-time of constant sectional curvature -1. Then the first fundamental form of f_{ads} is given by

$$\frac{\cosh 2\alpha - \cosh 2v}{2\sinh^2\alpha} du^2 + dv^2.$$

In particular, f_{ads} is space-like (resp. time-like) if $\cosh 2\alpha > \cosh 2v$ (resp. $\cosh 2\alpha < \cosh 2v$).

3. Properties of points where surfaces change type

In this section, we shall investigate the properties of functions $t = f(x,y)$ whose graphs induce mixed type surfaces in $\boldsymbol{R}^3_1$ with bounded mean curvature.

Definition 3.1 (cf. [3, Definition 2.3])**.** Let U be a domain in the xy-plane $\boldsymbol{R}^2$, and $f : U \to \boldsymbol{R}$ a C^∞-function. We set

$$B := 1 - f_x^2 - f_y^2.$$

A point $p \in U$ is called a *non-degenerate point of type change* if

$$B(p) = 0, \qquad \nabla B(p) \neq 0 \tag{3.1}$$

hold, where $\nabla B := (B_x, B_y)$.

By definition, the first fundamental form of the graph of f is degenerate at a non-degenerate point of type change. We set

$$A := (f_x^2-1)f_{yy} - 2f_xf_yf_{xy} + (f_y^2-1)f_{xx}.$$

Then the functions A, B can be considered as a special case of the functions (cf. (2.1) and (2.4)) α, β by setting $(u,v) = (x,y)$. By (1.3), we have

$$H = \frac{A}{2|B|^{3/2}}. \tag{3.2}$$

Proposition 3.2 (cf. Proposition 2.4 in [3])**.** *Suppose that the mean curvature function of the graph of f is bounded. Let $p \in U$ be a point satisfying $B(p) = 0$. Then the following two assertions are equivalent:*

1. *the point p is a non-degenerate point of type change.*
2. *p is a dually regular point in the sense of* [7]*, that is, p is a point where $f_{xx}(p)f_{yy}(p) - f_{xy}(p)^2 \neq 0$.*

Proof. The proof is almost parallel to that of Proposition 2.4 in [3]. It holds that

$$\nabla B = \mathrm{Hess}(f)\begin{pmatrix} f_x \\ f_y \end{pmatrix}, \qquad \mathrm{Hess}(f) := \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}. \tag{3.3}$$

Now suppose that (2) holds. Then $\mathrm{Hess}(f)$ is a regular matrix at p. Since $B(p) = 0$, $(f_x, f_y) \neq 0$ at p. Thus, (3.3) implies that $\nabla B \neq 0$ at p, that is, (1) holds.

We next suppose on the contrary that (2) does not hold. By a suitable linear coordinate change of (x, y), we may assume without loss of generality that $f_{xy}(p) = 0$. Then either $f_{xx}(p) = 0$ or $f_{yy}(p) = 0$. By (3.2), and Theorem 1.1, we have $A(p) = 0$. This with $B(p) = 0$ and $f_{xy}(p) = 0$ implies that

$$f_x(p)^2 f_{xx}(p) + f_y(p)^2 f_{yy}(p) = 0.$$

This with $f_{xx}(p) = 0$ or $f_{yy}(p) = 0$ implies that

$$\mathrm{Hess}(f)\begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} f_x(p) f_{xx}(p) \\ f_y(p) f_{yy}(p) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So (1) does not hold. □

A regular curve $\Gamma : (a, b) \to \boldsymbol{R}^3_1$ is called *null* or *isotropic* if $\dot{\Gamma}(t) := d\Gamma(t)/dt$ is a light-like vector for each $t \in (a, b)$.

Definition 3.3. A null curve $\Gamma : (a, b) \to \boldsymbol{R}^3_1$ is called non-degenerate at $t = c$ if $\dot{\Gamma}(c)$ and $\ddot{\Gamma}(c)$ are linearly independent. If $\Gamma(t)$ is non-degenerate for all $t \in (a, b)$, the curve Γ is called a *non-degenerate* null curve.

Let $p \in U$ be a non-degenerate point of type change. Then, by the implicit function theorem, there exists a regular curve $\gamma : (-\varepsilon, \varepsilon) \to U$ such that $B \circ \gamma(t) = 0$ and $\gamma(0) = p$, where ε is a positive number. We call this curve γ *the characteristic curve* of type change. The following assertion is a generalization of [3, Proposition 2.5] for zero-mean curvature surfaces.

Proposition 3.4. *Suppose that the graph $t = f(x, y)$ over a domain U has bounded mean curvature function. If the graph changes type along a regular curve $\gamma(t)$ ($|t| < \varepsilon$) such that $f \circ \gamma(t)$ is a non-degenerate null curve in $\boldsymbol{R}^3_1$, then $\gamma(t)$ consists of non-degenerate points of type change.*

Proof. The proof is completely parallel to that of [3, Proposition 2.5]. □

The converse assertion is given as follows, which is a generalization of [3, Proposition 2.6] for zero-mean curvature surfaces.

Proposition 3.5. *Suppose that the graph $t = f(x, y)$ over a domain U has bounded mean curvature function. Let $p \in U$ be a non-degenerate point of type change and $\gamma(t)$ ($|t| < \varepsilon$) the characteristic curve of type change such that $\gamma(0) = p$. Then $f \circ \gamma(t)$ is a non-degenerate null curve.*

Proof. Using the fact that $A(\gamma(t)) = 0$ holds, the proof of this assertion is completely parallel to that of [3, Proposition 2.6]. □

Moreover, the following assertion holds:

Proposition 3.6. *Let $t = f(x, y)$ be a real analytic function over the domain U which gives a graph with bounded mean curvature function. Suppose that the zeros of $B(x, y)$ are all non-degenerate points of type change. Then, the mean curvature vector $H\nu$ can be analytically extended to all of U.*

Proof. Let $p \in U$ be a non-degenerate point of type change. Then we can take a real analytic local coordinate system (u, v) centered at p such that the u-axis is the characteristic curve of type change. By the condition $\nabla B(u, 0) \neq (0, 0)$ (cf. (3.1)), there exists a real analytic function $b(u, v)$ defined near the u-axis such that $B(u, v) = vb(u, v)$ and $b(u, 0) \neq 0$. On the other hand, Theorem 1.1 yields that there exists a real analytic function $a(u, v)$ defined near the u-axis such that

$$A(u, v) = v^2 a(u, v). \tag{3.4}$$

By (3.2), we have

$$H(u, v) = \frac{\sqrt{|v|}a(u, v)}{2|b(u, v)|^{3/2}}.$$

By (1.2), we have that

$$H\nu = \frac{\sqrt{|v|}a(u, v)}{2|b(u, v)|^{3/2}} \frac{1}{\sqrt{|v||b(u, v)|}}(1, f_x, f_y) = \frac{a(u, v)}{2b(u, v)^2}(1, f_x, f_y), \tag{3.5}$$

proving the assertion. □

Finally, we prove Theorem 1.2 in the introduction:

Proof of Theorem 1.2. Let $f : \boldsymbol{R}^2 \to \boldsymbol{R}$ be a real analytic function whose graph gives a zero-mean curvature surface, with function $B := 1 - f_x^2 - f_y^2$ satisfying $\nabla B \neq (0, 0)$ if $B = 0$. Take a real analytic function $\psi : \boldsymbol{R} \to \boldsymbol{R}$ such that

$$\psi(0) = \psi'(0) = \psi''(0) = 0. \tag{3.6}$$

We then set

$$g(x, y) := f(x, y) + \psi(B(x, y)),$$

and

$$\tilde{B} := 1 - g_x^2 - g_y^2.$$

Since

$$g_x = f_x + \psi'(B)B_x, \quad g_y = f_y + \psi'(B)B_y,$$

we have that

$$\tilde{B} = B - 2\psi'(B)(f_x B_x + f_y B_y) - \psi'(B)^2(B_x^2 + B_y^2). \tag{3.7}$$

Here, the relation $C_1 \equiv C_2 \mod B$ for two real analytic functions $C_i(x, y)$ $(i = 1, 2)$ means that $(C_1 - C_2)/B$ is a real analytic function on $\boldsymbol{R}^2$. Since $\psi'(B) \equiv 0 \mod B$, $\tilde{B}$ can be divided by B. Thus, to show the mean curvature vector field can be smoothly extended across the set $B = 0$, it is sufficient to show that

$$\tilde{A} := (g_y^2 - 1)g_{xx} - 2g_x g_y g_{xy} + (g_x^2 - 1)g_{yy}$$

can be divided by B^2. Since

$$\begin{aligned} g_{xx} &= f_{xx} + \psi''(B)B_x^2 + \psi'(B)B_{xx}, \\ g_{xy} &= f_{xy} + \psi''(B)B_x B_y + \psi'(B)B_{xy}, \\ g_{yy} &= f_{yy} + \psi''(B)B_y^2 + \psi'(B)B_{yy}, \end{aligned}$$

the fact that $A = 0$ yields

$$\tilde{A} \equiv \psi''(B)\Gamma + \psi'(B)\Delta \mod B^3, \tag{3.8}$$

where

$$\begin{aligned}\Gamma &:= (f_y^2-1)B_x^2 - 2f_xf_yB_xB_y + (f_x^2-1)B_y^2,\\ \Delta &:= 2(B_xf_xf_{yy} - B_xf_{xy}f_y - B_yf_xf_{xy} + B_yf_{xx}f_y)\\ &\qquad + B_{xx}\left(f_y^2-1\right) - 2B_{xy}f_xf_y + B_{yy}\left(f_x^2-1\right).\end{aligned}$$

Since

$$\begin{aligned}\Gamma &= (-B-f_x^2)B_x^2 - 2f_xf_yB_xB_y + (-B-f_y^2)B_y^2\\ &= -B(B_x^2+B_y^2) - (f_xB_x+f_yB_y)^2\end{aligned}$$

and

$$\begin{aligned}f_xB_x+f_yB_y &= -2(f_x(f_xf_{xx}+f_yf_{xy}) + f_y(f_xf_{xy}+f_yf_{yy}))\\ &= 2A + 2B(f_{xx}+f_{yy}) = 2B(f_{xx}+f_{yy}),\end{aligned} \tag{3.9}$$

we have that

$$\Gamma \equiv -B(B_x^2+B_y^2) \mod B^2. \tag{3.10}$$

Since

$$\psi'(B) \equiv 0 \mod B^2, \qquad \psi''(B) \equiv 0 \mod B,$$

(3.8) and (3.10) yield that $\tilde{A}$ can be divided by $\tilde{B}^2$.

To give an explicit example, we consider the function $f_K(x,y) := x\tanh y$ given in Example 2.5. Then, we have

$$B(x,y) = (\cosh^2 y - x^2)\mathrm{sech}^4 y$$

and $x = \pm\cosh y$ give the characteristic curves of type change. We consider the new function

$$g(x,y) := x\tanh y + c\tanh^3(B(x,y)) \qquad (0 < c \le 1), \tag{3.11}$$

where c is a constant. Then the mean curvature vector field is real analytic along the set of type change $\Sigma_f := \{(\pm\cosh y, y)\,;\, y \in \boldsymbol{R}\}$.

By (3.10),

$$\Gamma \equiv -4B\mathrm{sech}^4 y \mod B^2$$

holds. By a straightforward calculation,

$$\Delta = 2(B+1)\mathrm{sech}^4 y$$

holds. Since $\psi(B) = c\tanh^3(B)$, we have

$$\psi'(B) \equiv 3cB^2, \quad \psi''(B) \equiv 6cB \mod B^3.$$

Thus, (3.8) yields that

$$\left.\frac{\tilde{A}}{\tilde{B}^2}\right|_{(x,y)=(\pm\cosh y, y)} = \left.\frac{\tilde{A}}{B^2}\right|_{(x,y)=(\pm\cosh y, y)} = \frac{-18c}{\cosh^4 y} \qquad (y \in \boldsymbol{R}), \tag{3.12}$$

which never vanishes on the set Σ_f.

To complete the proof, it is sufficient to show that $\tilde{B}/B$ has no zeros if c is sufficiently small. We shall now compute $\tilde{B}/B$ using (3.7). We set

$$\varphi(t) := \frac{\tanh t}{t}$$

which is a real analytic bounded function. We set

$$U := x\,\mathrm{sech}^2 y, \quad V := \mathrm{sech}\, y, \quad S := \mathrm{sech}(V^2 - U^2).$$

Here U is unbounded, but V, S are bounded on $\boldsymbol{R}^2$. By a straight-forward calculation, one can get that

$$\frac{\tilde{B}}{B} = 1 - 12cB\varphi(B)^2 S^2 (C_1 + C_2),$$

where

$$C_1 := 2U(U^2 - V^2)\tanh y,$$
$$C_2 := 3cB^2\varphi(B)^2 S^2 \biggl(U^2V^4 + (-2U^2 + V^2)^2 \tanh^2 y\biggr).$$

Since

$$\frac{\cosh(V^2 - U^2)}{\cosh(U^2)} = \cosh(V^2) - \sinh(V^2)\tanh(U^2),$$

using the fact that $|V| \leq 1$, we have

$$e^{-1} \leq \exp(-V^2) = \cosh(V^2) - \sinh(V^2) < \frac{\cosh(V^2 - U^2)}{\cosh(U^2)}.$$

In particular

$$S|U|^m = \frac{|U|^m}{\cosh(U^2)} \frac{\cosh(U^2)}{\cosh(V^2 - U^2)} < \frac{e|U|^m}{\cosh(U^2)}$$

is a bounded function for $m \geq 0$. Then we can write

$$\frac{\tilde{B}}{B} = 1 - 12c\varphi(B)^2 SB(SC_1 + SC_2).$$

Since $\tanh y$, $\varphi(B)$, and $SB = B\operatorname{sech} B$ are all bounded, there exists a positive constant m which does not depend on the choice of $c \in (0, 1]$ such that $\varphi(B)^2 SB(SC_1 + SC_2) < m$ holds for all $(x, y) \in \boldsymbol{R}^2$, and so

$$\left| \frac{\tilde{B}}{B} - 1 \right| < 12mc.$$

If $0 < c < 1/(12m)$, then the zero set of $\tilde{B}$ coincides with that of B, proving the assertion. □

Acknowledgements. The first, the fourth and the fifth authors thank Udo Hertrich-Jeromin and Kosuke Naokawa for fruitful conversations at TU-Wien. The authors thank Wayne Rossman for valuable comments.

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(Atsufumi Honda) National Institute of Technology, Miyakonojo College, 473-1, Yoshio-cho, Miyakonojo, Miyazaki 885-8567, Japan
E-mail address: `atsufumi@cc.miyakonojo-nct.ac.jp`

(Miyuki Koiso) Institute of Mathematics for Industry, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka, 819-0395, Japan
E-mail address: `koiso@math.kyushu-u.ac.jp`

(Masatoshi Kokubu) Department of Mathematics, School of Engineering, Tokyo Denki University, Tokyo 120-8551, Japan
E-mail address: `kokubu@cck.dendai.ac.jp`

(Masaaki Umehara) Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, 2-12-1-W8-34, O-okayama Meguro-ku, Tokyo 152-8552, Japan
E-mail address: `umehara@is.titech.ac.jp`

(Kotaro Yamada) Department of Mathematics, Tokyo Institute of Technology, O-okayama, Meguro, Tokyo 152-8551, Japan
E-mail address: `kotaro@math.titech.ac.jp`